Some conjectures and results about multizeta values for $\mathbb{F}_q[t]$

by

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Statement by Author

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Chapter 1
Introduction

1.1 Special values of classical zeta

The Riemann zeta function is defined by

$$\zeta_{\mathbb{Q}}(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p\,\text{prime}} (1 - p^{-s})^{-1},$$

where $s \in \mathbb{C}$ with $\Re s > 1$. We can analytically continue $\zeta_{\mathbb{Q}}(s)$ to a meromorphic function on $\mathbb{C}$ with a pole of order 1 and residue 1 at 1. There is a rich *special values* theory associated to $\zeta_{\mathbb{Q}}(s)$, which is intimately connected to Bernoulli numbers, B_n. If $n \geq 0$ we have

$$\zeta_{\mathbb{Q}}(-n) = -\frac{B_{n+1}}{n+1}.$$

Consequently, if $n \geq 1$, $\zeta_{\mathbb{Q}}(-2n) = 0$. Such zeros are called trivial zeros and they are simple zeros. With respect to the non-trivial zeros, the well known Riemann hypothesis says that the non-trivial zeros of $\zeta_{\mathbb{Q}}(s)$ lie on the line $\Re s = \frac{1}{2}$.

All the zeros found so far have turned out to be simple zeros, so nowadays simplicity of zeros is also conjectured. The Riemann Hypothesis has many interesting consequences, e.g., in the distribution of primes. For $m = 2k$, $k > 0$ an integer we have Euler's Theorem

$$\zeta_{\mathbb{Q}}(m) = -\frac{B_m (2\pi i)^m}{2(m!)}.$$

There is no simple formula for $\zeta_{\mathbb{Q}}(2k+1)$ analogous to the previous one. It is not known whether $\zeta_{\mathbb{Q}}(2k+1)$ is rational or irrational, except for $k = 1$ when it is irrational. Also, divisibilities of B_m by primes p are closely related to information on components of the ideal class group of cyclotomic extensions $\mathbb{Q}(\mu_p)$, where μ_p is a primitive pth root of unity. For example, see Herbrand-Ribet Theorem in [Was97].

More generally the *Dedekind zeta function* ζ_K of a number field K (a finite extension of $\mathbb{Q}$) is defined, for $s \in \mathbb{C}$ with $\Re s > 1$, by

$$\zeta_K(s) := \sum_{\mathcal{I}} \mathrm{N}(\mathcal{I})^{-s} = \prod_{\mathcal{P}} (1 - \mathrm{N}(\mathcal{P})^{-s})^{-1},$$

where the sum is taken over all non-zero ideals of $\mathcal{O}_K$ (ring of integers of $K/\mathbb{Z}$). Here $\mathrm{N}(\mathcal{I}) = |\mathcal{O}_K/\mathcal{I}|$ is the norm of the ideal $\mathcal{I}$, and $\mathcal{P}$ runs through the prime ideals $\mathcal{P}$ of $\mathcal{O}_K$. Notice that for $K = \mathbb{Q}$, $\zeta_K = \zeta_{\mathbb{Q}}$ since $\mathrm{N}(n\mathbb{Z}) = |\mathbb{Z}/n\mathbb{Z}| = n$. This function has a simple functional equation connecting $\zeta_K(s)$ to $\zeta_K(1-s)$. Let r_1 the number of embeddings of K in $\mathbb{R}$ and r_2 half the number of non-real embeddings of K in $\mathbb{C}$. For $s > 1$, it is clear that there are no zeros and hence analyzing the poles of the gamma factors in the functional equation, we can see that, at negative integers, the zeta function vanish to order $r_1 + r_2$ (r_2 respectively), if s is even (odd). In addition, for s a positive even integer, $\zeta_K(s)/(2\pi i)^{r_1 s} \in \mathbb{Q}$, if K is totally real. Furthermore, we have the *analytic class number formula*,

$$\lim_{s \to 1+} (s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2} R}{m\sqrt{|D|}} h,$$

where h, D, and R denote the class number, the discriminant, and the regulator of the number field K, and m is the number of roots of unity contained in K.

In general, orders of vanishing and special (leading) values encode a lot of interesting arithmetic information.

1.2 Two kinds of zeta in function fields

For a function field K over the finite field of constants $\mathbb{F}_q$, $q = p^n$, we describe the *Artin-Weil zeta function*. For a divisor $\mathcal{D}$, we put